A new approach to problems of shock dynamics
Part I Two-dimensional problems

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SUMMARY

In this paper, two-dimensional problems of the diffraction and stability of shock waves are investigated using an approximate theory in which disturbances to the flow are treated as a wave propagation on the shocks. These waves carry changes in the slope and the Mach number of the shock. The equations governing the wave propagation are analogous in every way to the non-linear equations for plane waves in gas dynamics, and their solutions can be deduced by the same mathematical techniques. Since the propagation speed of the waves is found to be an increasing function of Mach number, waves carrying an increase in Mach number will eventually break and form what we may call a 'shock', corresponding to the breaking of a compression wave into a shock in the ordinary plane wave case. Such a 'shock' moving on the shock is called a *shock–shock*. The shock–shock is a discontinuity in Mach number and shock slope, and it must be fitted in to satisfy the appropriate relations between these discontinuities and its speed. The waves moving on the shock are interpreted as the trace of cylindrical sound waves in the flow behind the shock. In particular a shock–shock is the trace of a genuine shock in the flow behind, and thus corresponds to Mach reflection.

The general theory of the wave propagation is set out in §2. The subsequent sections contain applications of the theory to specific problems, including the motion of a shock along a curved wall, diffraction by a wedge, stability of plane shocks and the instability of a converging cylindrical shock.

1. INTRODUCTION

In this paper a relatively simple approximate method is developed for treating problems of the diffraction and stability of shock waves. The theory can be formulated without reference to any specific problem and it is convenient to give the basic ideas before discussing the applications. Only two-dimensional problems are considered in this first part; the extension to other cases is given in Part II.

We start by considering the set of curves formed by the successive positions of a curved shock as it moves forward through a uniform medium, and we introduce the orthogonal trajectories of this set of curves. These orthogonal trajectories will be called 'rays'. In figure 1 the positions of a shock moving from left to right are shown as full lines and the rays are shown as broken lines. This network of shock positions and rays may be used as a basis for orthogonal coordinates in the plane, and accordingly coordinates (α, β) are introduced such that the shock positions are the curves α = constant, and the rays are β = constant. A suitable α coordinate would be the time t at which the shock occupies that position, but we modify this slightly and

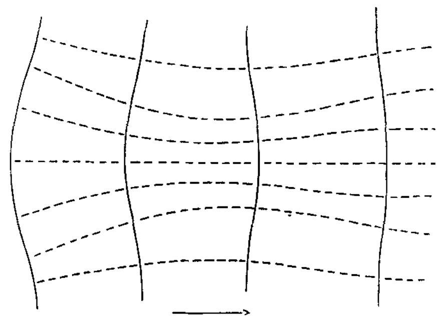

Figure 1. Sketch showing the successive positions of a curved shock ; the full and broken lines represent the shock positions and the rays, respectively.

take $\alpha = a_0 t$ where a_0 is the sound speed in the uniform gas ahead of the shock. Then, the distance along a ray between the shock positions given by α and $\alpha + d\alpha$ is $M(\alpha, \beta)\, d\alpha$ where M is the Mach number of the shock at (α, β). If we let $A(\alpha, \beta)\, d\beta$ be the corresponding distance between the rays β and $\beta + d\beta$, then it can be shown that, for purely geometrical reasons, M and A must satisfy the differential relation

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{M}\frac{\partial A}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{A}\frac{\partial M}{\partial \beta}\right) = 0. \tag{1}$$

An elementary proof of this will be given in § 2 but from a more sophisticated point of view it is the condition for the space to be flat; the curvature tensor can be expressed in terms of M and A, and (1) is the only component which is not identically zero. This alternative derivation is given in some detail in Part II since it is the neatest way of obtaining the relations corresponding to (1) for three dimensions.

Now, if we can find a second relation between M and A, we have an explicit equation for the Mach number of the shock as a function of (α, β); from this function, the shock position can be determined for all times. The second relation must come from the dynamics of the motion and strictly requires a solution of the equations of motion for the flow behind the shock, subject to the Rankine–Hugoniot relations across the shock and boundary conditions at solid walls, etc. Of course this is the original problem. But the above approach suggests a simple approximate procedure.

To some extent, the propagation of the shock between any two neighbouring rays can be treated as if the rays were solid walls. This would be exactly true if the rays were particle paths, but the most we can say is that immediately behind the shock the particles move normal to the shock, i.e. in the ray direction. However, we assume that the later divergence of the rays and the particle paths is not important and accept this similarity to propagation in a channel. Now, for a shock moving down a channel, if the modifications to the shock arise only from changes in channel area, the Mach number is a function of the area. Taking this over to propagation in the channel formed by neighbouring rays, we have the functional dependence

$$A = A(M) \tag{2}$$

as the second relation between A and M. This is the only assumption in the theory and we can proceed from (1) and (2) without further approximation. Qualitatively, the results are independent of the precise choice of (2) provided only that A is a decreasing function of M. For numerical results we make use of the function $A(M)$ obtained and used recently by R. F. Chisnell (1957) for the motion of shocks down converging channels. In an earlier paper, Chester (1954) found that for a small change dA in channel area the corresponding change in Mach number is given by

$$\frac{dA}{A} = \frac{-2M\,dM}{(M^2-1)\,K(M)}, \tag{3}$$

where $K(M)$ is a slowly varying function, decreasing from 0·5 at $M = 1$ to 0·3941 (for $\gamma = 1{\cdot}4$) as $M \to \infty$. Chisnell suggested that the integrated form of (3) should give a good approximation for a channel of slowly varying cross-section; his work on the converging cylindrical shock confirms this view. On integration, (3) gives

$$A = kf(M), \qquad f(M) = \exp\left\{-\int \frac{2M\,dM}{(M^2-1)\,K(M)}\right\}, \tag{4}$$

where k is an arbitrary constant which may be different for each channel, i.e. $k = k(\beta)$. In many cases the shock is initially of constant strength with M and A taking constant values M_0 and A_0. Then, for each channel k has the same value $A_0/f(M_0)$ and it can be absorbed into $f(M)$. Even if M_0 is not constant on $\alpha = 0$, the k can be suppressed by a careful choice of the coordinate β; it is only necessary to arrange that $A_0 = f(M_0)$. For the time being we assume this is done. Later, however, we shall see that the dependence of k on β can arise in a less trivial way.

The function $K(M)$ is given by

$$K(M) = 2\left[\left(1+\frac{2}{\gamma+1}\frac{1-\mu^2}{\mu}\right)(2\mu+1+M^{-2})\right]^{-1}, \qquad \mu^2 = \frac{(\gamma-1)M^2+2}{2\gamma M^2-(\gamma-1)},$$

and its graph is shown in figure 2. Chisnell has shown that the integral in (4) can be evaluated explicitly; a graph of $\log_{10} f(M)$ is given in figure 3.

With $A = A(M)$, (1) becomes a second-order hyperbolic equation for M with two independent variables. In the exact formulation of the problem there are three independent variables; much of the mathematical simplification is in the reduction of the number of independent variables. The solution represents waves moving in each direction on the shock face.

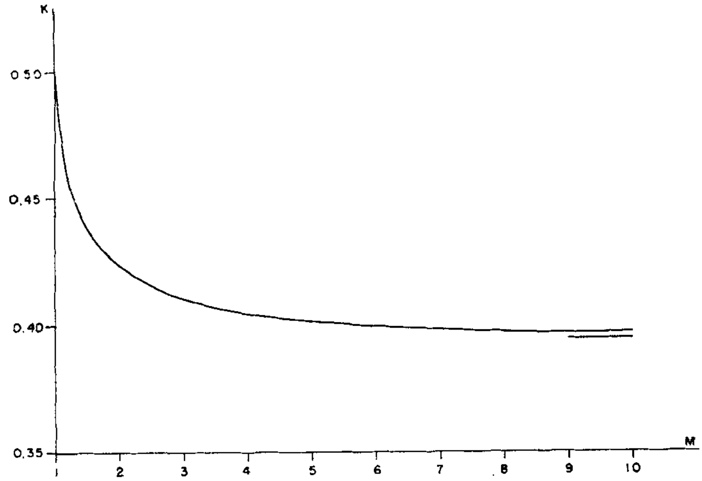

Figure 2. Graph of Chester's function $K(M)$.

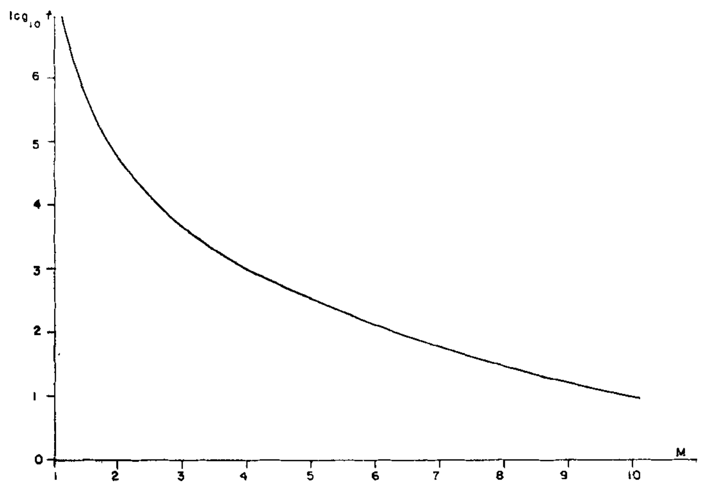

Figure 3. Graph of the function $\log_{10} f(M)$ given by equation (4).

Since equation (1) expresses a kinematic relation, these waves are a further example of 'kinematic waves' in the sense used by Lighthill & Whitham (1955). In fact, this example extends the idea of kinematic waves since the examples studied previously involved only first-order differential equations and, consequently, the wave propagation was in one direction only. It turns out that there is a close analogy between the waves travelling on the shock and sound waves of finite amplitude in one-dimensional gas dynamics, and all the mathematical methods used in that field are available for the present case*. The propagation speed for a wave, i.e. rate of change of β with respect to α, is an increasing function of M so that waves carrying a decrease in the value of M spread out like expansion waves in gas dynamics; similarly, the profile of a wave carrying an increase in M steepens like a compression wave. In the latter case, the wave will eventually break and then to complete the solution a discontinuity in Mach number and in shock slope must be fitted in. This discontinuity is analogous to the shock wave of gas dynamics and it must be included in such a way that the appropriate 'shock relations' connecting its speed with the jumps in M, etc., are satisfied.

Since all the features of gas dynamics arise in the study of the wave motion on the shock, it is desirable to use the same terminology because it automatically conjures up the right ideas. However, the discontinuous wave would then be called a 'shock' and unless the word is qualified in some way there may be confusion with its direct use for the true shock. To avoid this, we shall always refer to these 'shocks' moving on the true shock as *shock-shocks.*

The waves on the shock are interpreted as the trace of cylindrical waves which are spreading out in the flow behind the shock. Thus, an 'expansion wave' on the shock is the trace of a cylindrical expansion wave in the flow; a 'compression wave' is interpreted similarly. Therefore, a shock-shock must be the trace of a genuine shock produced in the flow behind the main shock. Thus, it arises in Mach reflection and represents the three shock intersection characteristic of that phenomenon. This feature of the theory is particularly valuable since Mach reflection is of great importance in diffraction theory and is difficult to treat theoretically in all but the simplest cases. All the mathematical details of the wave motion, including a discussion of the appropriate conditions relating quantities on the two sides of a shock-shock, are given in § 2.

It is useful to think of the theory as the generalization to shock waves of the theory of geometrical acoustics. Geometrical acoustics applies to weak sound pulses and is closely related to the special case of weak shocks in the present theory. But there are differences, connected with the linearization in geometrical acoustics, which can be important. First, in geometrical acoustics (1) is replaced by the following determination of A, which is independent of M. If the shock is very weak, its velocity at different points varies only slightly and is always close to the sound speed a_0. In accordance with the linear theory of acoustics, these small variations are neglected and the propagation speed takes the constant value a_0. Hence, the rays are straight lines, determined once and for all as the normals to a given initial shock position, and the area A of any ray tube can be calculated.

* It is assumed in this paper that the reader is familiar with the theory of one-dimensional gas dynamics as presented for example, in Courant & Freidrichs (1948).

The variation of the strength of the shock is then deduced from the assumption that the energy flux down any ray tube remains constant along the tube. Since the energy flux is proportional to the square of the amplitude of the wave multiplied by A, it follows from this argument that the strength is proportional to $A^{-1/2}$. For a weak shock, the strength is proportional to $M-1$, so that

$$M-1 \propto A^{-1/2}.$$

We may note that this agrees with the Chisnell formula as $M \to 1$, since $K = 0{\cdot}5$ in (4). Even for weak shocks, however, geometrical acoustics is inadequate in certain cases and the general formulation must be used. This is due to the linearization which is introduced by assuming that the propagation speed can be approximated by a_0; although the variations of the speed are small they cannot always be neglected. Consider, for example, a shock which is initially concave forward as in figure 4. The

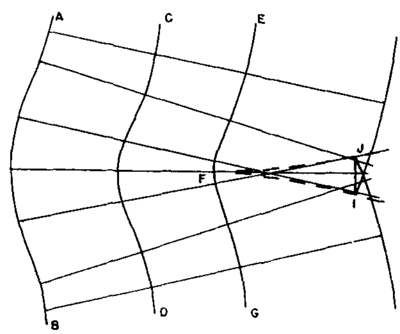

Figure 4. Shock positions and rays according to geometrical acoustics ; AB, CD, EFG, $HIJK$ represent successive positions of the shock.

normals to the initial surface form an envelope, called a 'caustic', at which $A \to 0$, and consequently geometrical acoustics predicts infinite strength. Moreover, beyond the caustic the position of the shock as calculated by geometrical acoustics folds over itself. To obtain the true behaviour it is absolutely essential for the small variations in speed and the corresponding distortion of the rays to be included. Then nothing very remarkable happens. As the strength increases in the concave part, the shock moves faster there and tends to smooth out the shape; the rays are pushed away from each other and now avoid any intersection. In fact the region of the shock which was originally concave overshoots and becomes convex. Then the velocity decreases until the shock ultimately smooths out into a plane. The true picture takes the form sketched in figure 1; the appropriate mathematical discussion using equations (1) and (2) is given in §4. The above argument is also a rough explanation of the observed result that plane shocks are very stable; the theory of this paper puts the argument in mathematical form. For further discussion of the differences between weak shocks and sound pulses, reference may be made to the author's paper on weak shock waves (Whitham 1956).

Turning now to the question of applications, it is clear from the outset that the theory can only be expected to apply to certain types of problem. The simplicity of the method is achieved by avoiding a detailed discussion of the flow behind the shock through the assumption that A is a function of M. Clearly, the flow cannot be forced into such a subsidiary role in general. It would not be feasible, for example, in explosion problems in which the propagation towards the shock of disturbances originating far behind it must be the prime consideration. But, for the diffraction of a uniform plane shock by an obstacle, the disturbance originates at the shock so that it is more understandable (although still surprising, perhaps) that the discussion can be limited to a neighbourhood of the shock. The accuracy of the results is a measure of how far this is possible. Again, according to the rough argument given in the previous paragraph, stability is largely a question of local adjustment of the flow near the shock and can be included in the applications.

Mathematically, the easiest problem to solve is the diffraction of a shock moving along a non-uniform wall. If the wall always turns away from the flow region, an 'expansion wave' originating at the wall moves out along the shock and the solution is a 'simple wave' in the terminology of gas dynamics. If the wall turns towards the fluid a 'compression wave' is sent along the shock and eventually a shock–shock is formed. The appropriate relations connecting quantities on the two sides of the shock–shock must be introduced in this case. A special case is diffraction by a wedge in which the solution is given entirely by a shock–shock separating two uniform regions; this is the familiar Mach reflection. It must be pointed out immediately that the present theory does not throw any further light on the difficulties in the conventional solution for conditions at a three shock intersection. Its main contribution is to give a method for treating variations in the three shock configuration which would be caused, for example, by further curvature of the wall of the wedge. The special case of a small corner (both expansive and compressive) offers an opportunity of checking the results with those of the linear theories of Lighthill (1949) and Ting & Ludloff (1952). The comparison shows that the approximate method is most suitable for strong shocks with Mach number greater than about 2, and must be used with care for weaker ones. The predicted changes in Mach number are in good agreement with Lighthill's values for all strengths, but for very weak shocks the geometry of the disturbed flow is not given accurately. The details of the application to diffraction problems are set out in § 3.

In § 4, the stability of plane shocks is considered and the results are compared with those found by Blackburn (1953) and Freeman (1955) working with Lighthill's linear theory. It is seen that the stability predicted by the theory of this paper is achieved by a different process. The essential mechanism here is the non-linearity of the waves and the dissipation of their energy in shock–shocks (in the same way that shock waves dissipate energy in ordinary sound waves). In the linearized theory of Blackburn and

Freeman, the stability depends on damping of the waves through some diffusion process (although, of course, this does not appear explicitly in their work). The non-linear theory predicts that a sinusoidal disturbance on a nearly plane shock will decay with time like t^{-1}, whereas Blackburn and Freeman predict decay as $t^{-1/2}$ for strong shocks and $t^{-3/2}$ otherwise. It is not clear at this stage which of the two effects is more important or how the two sets of results are related.

Finally, in § 5, we continue Butler's theory (Butler 1955) of the instability of a converging cylindrical shock. This work is closely similar to Butler's investigation and the results are the same. The advantage of the derivation given here is that we linearize the non-linear equation for M by the hodograph transformation and no approximation is made. Although Butler does not obtain or use (1), his method is essentially equivalent to assuming small perturbations about the symmetric solution in equation (1) and retaining only the linear terms in the perturbations. The method given in § 5 is free from questions of the validity of the linear approximations and confirms Butler's results. It also offers an opportunity to show how the hodograph transformation may be used in this work.

The relations corresponding to (1) in three-dimensional problems are obtained in Part II. In particular, they are specialized to axisymmetrical problems. Then, as might be expected, the waves on the shock are analogous to cylindrical waves in gas dynamics. For example, diffraction of a plane shock by a cone is analogous to the problem in gas dynamics of a cylindrical piston expanding with uniform velocity; the same method of solution can be used.

2. General theory of the wave motion on the shock

To establish the geometrical relationship between M and A, consider the curvilinear quadrilateral $PQRS$ with vertices (α, β), $(\alpha+\delta\alpha, \beta)$, $(\alpha+\delta\alpha, \beta+\delta\beta)$, $(\alpha, \beta+\delta\beta)$ respectively (see figure 5). Let $\theta(\alpha, \beta)$ be the angle made by the ray with a fixed direction. Since the sides PS and QR are $A\delta\beta$ and $\{A+(\partial A/\partial\alpha)\delta\alpha\}\delta\beta$, respectively, and the distance between them is $M\delta\alpha$, the change in ray inclination from P to S is

$$\delta\theta = \frac{QR-PS}{PQ} = \frac{1}{M}\frac{\partial A}{\partial \alpha}\,\delta\beta.$$

Hence

$$\frac{\partial\theta}{\partial\beta} = \frac{1}{M}\frac{\partial A}{\partial\alpha}. \tag{5}$$

Since the inclination of the β-curves is $\frac{1}{2}\pi+\theta$, a similar argument shows that

$$\frac{\partial\theta}{\partial\alpha} = -\frac{1}{A}\frac{\partial M}{\partial\beta}. \tag{6}$$

If θ is eliminated from (5) and (6), equation (1) quoted in the Introduction is obtained. It is convenient, however, to work with the two first-order equations (5) and (6) instead of the second-order equation (1).

We now assume in (5) and (6) that $A = A(M)$ where $A'(M) < 0$. Then

$$\frac{\partial \theta}{\partial \beta} - \frac{A'(M)}{M}\frac{\partial M}{\partial \alpha} = 0, \tag{7}$$

$$\frac{\partial \theta}{\partial \alpha} + \frac{1}{A(M)}\frac{\partial M}{\partial \beta} = 0. \tag{8}$$

As noted in the last section, these equations are like the equations of non-linear sound waves and can be treated in the same way. Once the functions $M(\alpha,\beta)$ and $\theta(\alpha,\beta)$ have been found, the coordinates (α,β) may be related to the Cartesian coordinates (x,y) through the relations $y = \int M \sin\theta \, d\alpha$,

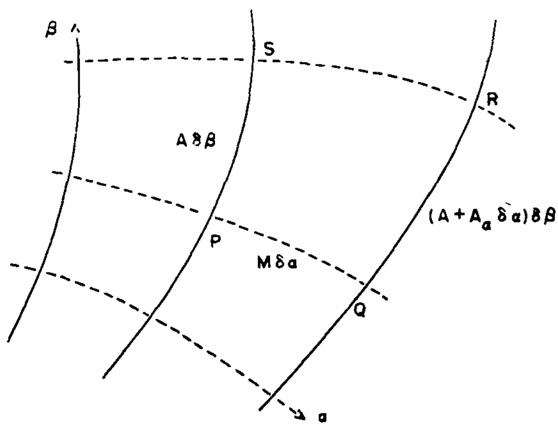

Figure 5. Neighbouring α and β curves in a region of continuous change in M and θ.

$x = \int M \cos\theta \, d\alpha$ obtained by integrating along a ray. The wave property is obtained by writing (7) and (8) in characteristic form, in which only derivatives in one direction appear. The characteristic form is

$$\left(\frac{\partial}{\partial \alpha} \pm c\frac{\partial}{\partial \beta}\right)\left(\theta \pm \int \frac{dM}{Ac}\right) = 0, \tag{9}$$

where c is the function of M given by

$$c = \sqrt{\frac{-M}{AA'}} = \sqrt{\frac{-d(M^2)}{d(A^2)}}; \tag{10}$$

it is easily verified that these equations are completely equivalent to the original ones. They show that

$$\theta + \int \frac{dM}{Ac} = \text{constant on } \frac{d\beta}{d\alpha} = c, \tag{11}$$

i.e. on a wave moving in the direction of increasing β with speed c, and

$$\theta - \int \frac{dM}{Ac} = \text{constant on } \frac{d\beta}{d\alpha} = -c, \tag{12}$$

i.e. on a wave moving in the direction of decreasing β with speed c. The expressions $\theta \pm \int dM/Ac$ in (11) and (12) correspond to the Riemann invariants of gas dynamics.

In the special case of a simple wave moving in the direction of β increasing,

$$\theta - \int \frac{dM}{Ac} = \text{constant } \textit{everywhere}; \tag{13}$$

hence, from (11), θ and M must be individually constant on each characteristic curve $d\beta/d\alpha = c(M)$, and this curve is a straight line in the (α, β) plane. The solution can then be determined completely in terms of appropriate boundary values. For example, let us suppose that θ is a given function $\theta_w(\alpha)$ on $\beta = 0$ and that initially the shock is undisturbed with $\theta = 0$, $M = M_0$. (This example arises in the next section for the motion of a shock along a wall.) Using the initial conditions, (13) determines the relation between θ and M, and in particular shows that the value M_w of M on $\beta = 0$ is given immediately in terms of θ_w by

$$\theta_w = \int_{M_0}^{M_w} \frac{dM}{Ac} \tag{14}$$

In the (α, β) plane, the slope of a characteristic starting on $\beta = 0$ at $\alpha = \alpha_w$ is $c(M_w(\alpha_w))$. Hence, since the characteristics are straight lines on which θ and M are constant, we have

$$\theta = \theta_w(\alpha_w), \qquad M = M_w(\alpha_w) \tag{15}$$

on

$$\beta = (\alpha - \alpha_w)\, c(M_w). \tag{16}$$

Since M_w is a known function of α_w, equation (16) determines α_w as a function of α and β; then (15) gives θ and M at (α, β).

In the general case, however, waves propagate in both directions; then (11) and (12) are the basic equations for the well known numerical method of characteristics.

To complete the theory we must consider the propagation of shock–shocks, i.e. discontinuities of M and θ. First, a simple theory is given which is suitable when the changes in M and θ are not too large; then, we shall see how this should be modified in other cases. Consider the neighbourhood of the discontinuity in two successive positions of the shock as shown in figure 6 (the rays through the corners are shown as broken lines). Let the difference in the α coordinates for the two shock positions be $\Delta\alpha$ and the difference in the β coordinates of the rays be $\Delta\beta$, and let subscripts 0 and 1 be used for values ahead of and behind the discontinuity, respectively. Then, in figure 6, $PQ = M_1\,\Delta\alpha$, $QR = A_1\,\Delta\beta$, $SR = M_0\,\Delta\alpha$, $PS = A_0\,\Delta\beta$. Expressing the distance PR in two alternative ways, we have

$$(A_0\,\Delta\beta)^2 + (M_0\,\Delta\alpha)^2 = (M_1\,\Delta\alpha)^2 + (A_1\,\Delta\beta)^2.$$

But, the ratio $\Delta\beta/\Delta\alpha$ is the shock–shock velocity C in the (α, β) coordinates; hence,

$$C^2 = -\frac{M_1^2 - M_0^2}{A_1^2 - A_0^2}. \tag{17}$$

If it is assumed that the functional relation (4) between A and M applies even for the sharp change in channel section at a shock–shock, the velocity C is determined by (17) in terms of M_0 and M_1. To deduce the corresponding change in θ, we note that angle $QPS = \theta_1 - \theta_0$ and this angle can be found from the geometry in figure 6. For,

$$\cot(\theta_1 - \theta_0) = \tan(RPQ + RPS)$$
$$= \left(\frac{A_1}{M_1} C + \frac{M_0}{A_0}\frac{1}{C}\right) \Big/ \left(1 - \frac{A_1 M_0}{M_1 A_0}\right).$$

Substituting from (17), we have

$$\cot(\theta_1 - \theta_0) = \frac{A_1 M_1 + A_0 M_0}{(M_1 - M_0^2)^{1/2}(A_0^2 - A_1^2)^{1/2}}. \tag{18}$$

It is easily verified for very weak shock–shocks that the velocity given by (17) reduces to the velocity (10) as $M_1 \to M_0$, $A_1 \to A_0$, and that, for small changes in M and θ, (18) gives the same relation as (13).

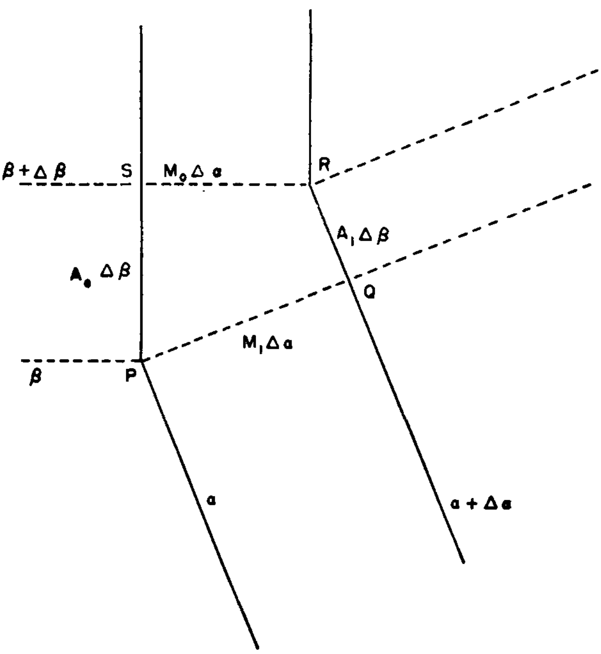

Figure 6. Neighbouring α and β curves at a shock–shock.

Although (17) and (18), together with (4), determine weak or even moderate shock–shocks with reasonable accuracy, we must consider further the question of stronger ones. We only go into this briefly to give the main ideas because considerably more labour would be involved in practical applications; this is not worthwhile until it is seen whether the original approximations prove satisfactory in practice. Nevertheless, the extensions do show up some valuable theoretical points. The limitation on the above relations for the shock–shocks is that for sufficiently large jumps in M, the dependence of A_1 on M_1 will not be given accurately by (4), which assumes that the channel section varies only slowly. Moreover, it is not just a question of establishing the correct formulae relating M and A at an abrupt change in channel section. In fact these formulae have been found by Laporte (1954). We must remember that there is a third shock in the flow behind the main shock, which must be considered in an accurate treatment of the conditions across a shock–shock. It does not invalidate the relations (17) and (18), but in general we cannot use the simple channel formula for the dependence of A_1 on M_1. Of course, it depends on the magnitude of the discontinuities; if $M_1 - M_0$ is not too large, the third shock will be weak and the channel formula can give a reasonable approximation.

The limitation can be seen in another way. Suppose we accept the approximate shock–shock relations so that $M_1 - M_0$ and the velocity C are determined in terms of the angle change $\theta_1 - \theta_0$ and the initial Mach number M_0. Then we can go on to fit in the third shock. But its strength must be chosen so that both the pressure and the stream deflection behind it are the same as behind the Mach shock. One of these conditions is sufficient to determine this third shock and we have a further condition still to satisfy. This corresponds to the fact that we have really assumed one condition too many in the original shock–shock conditions, namely, that the relation between A_1 and M_1 is known. If we relax this condition, the full three-shock theory will determine the relation in the course of fitting in the third shock. Thus, for more accurate conditions we may take the relations between M_1, A_1, θ_1 and M_0, A_0, θ_0 which are given by the conventional theory of the three-shock intersection; in particular, these will determine A_1.

With this more general determination of the shock–shock, let us go on to consider further the question of subsequent disturbances moving along the shock. In each ray channel, the *variations* in A and M are still related by (4), but in tracing back to determine the factor k at some known point, we can only go back as far as the shock–shock, even if the conditions ahead of it are uniform. Thus $k = A_1/f(M_1)$, where A_1 and M_1 are the values determined from the three-shock relations. Therefore k is a function of β which is to be determined in the course of the solution. Hence, in equations (5) and (6) the general form

$$A = k(\beta) f(M)$$

must be used. Fortunately, if the new variable $\int k(\beta)\, d\beta$ is taken instead of β, the equations are the same as (7) with $A(M)$ replaced by $f(M)$, and they can be solved in a similar way; we omit the details. The whole thing is rather like the question of entropy changes in gas dynamics. First of all one assumes that the pressure p and density ρ are functionally related ($p \propto \rho^\gamma$ usually) and this leads to simple waves and so on. But, then, since compression waves break, shocks have to be considered and they involve entropy changes so that p is no longer a function of ρ alone; behind the shock, the entropy is constant on each particle path. We have the analogous situation with A and M similar to p and ρ, and k playing a role similar to the entropy. Our simple theory of shock–shocks is rather like neglecting entropy changes at shocks in ordinary gas dynamics. This is known to give quite accurate results if the shock is not too strong, and we expect the same to be true here.

In the applications described in the following sections it will be necessary to make use of the properties of the functions $c(M)$ and $\int (Ac)^{-1}\, dM$. The functions are derived from equation (3), i.e.

$$\frac{dA}{A} = -\frac{2M}{(M^2-1)\,K(M)}\,dM,$$

and graphs of $K(M)$ and $A(M)$ have already been given in figures 2 and 3. From (10), we have

$$c(M) = (-M/AA')^{1/2} = \{\tfrac{1}{2}(M^2-1)\,K(M)\}^{1/2}/A. \qquad (19)$$

Now $c(M)$ is the propagation speed, i.e. the rate of change of β with respect to α of a wave in the (α, β) variables; hence the quantity Ac is the rate of

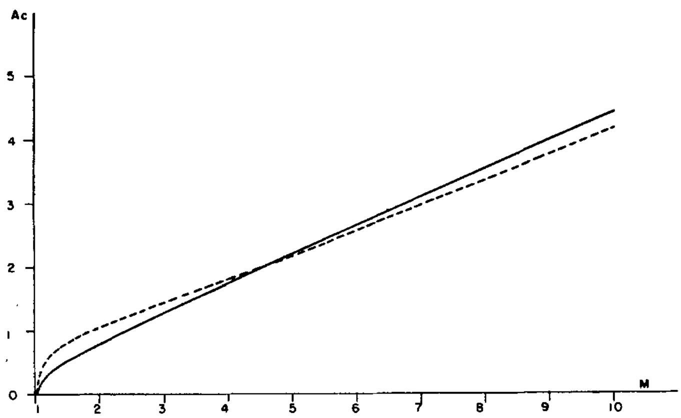

Figure 7. Graph of the propagation speed against M. The full line refers to the present theory (equation (19)) ; the broken line refers to the acoustic value given by (24).

change of *distance* with respect to α since $Ad\beta$ is the line element in the direction of β increasing. Thus Ac is a more useful as well as more convenient quantity to compute; the graph of Ac is shown in figure 7. The Riemann variable is given by

$$\int_1^M \frac{dM}{Ac} = \int_1^M \left\{\frac{2}{(M^2-1)\,K(M)}\right\}^{1/2} dM,$$

and its graph is shown in figure 8.

The approximations of the various functions in the special cases of weak and strong shocks are easily obtained using the results that $K \to 0{\cdot}5$

as $M \to 1$, and $K \to 0{\cdot}3941$ (for $\gamma = 1{\cdot}4$) as $M \to \infty$. The appropriate formulae are

$$\left.\begin{aligned} &K(M) \sim 0{\cdot}5, \qquad && Ac \sim (M-1)^{1/2}/2^{1/2}, \\ &\frac{A}{A_0} \sim \left(\frac{M_0-1}{M-1}\right)^2, && c \sim \frac{(M-1)^{5/2}}{2^{1/2}(M_0-1)^2 A_0}, \\ &\int_{M_0}^{M} \frac{dM}{Ac} \sim 2^{3/2}\{(M-1)^{1/2} - (M_0-1)^{1/2}\}, && \end{aligned}\right\} \text{ as } M \to 1, \tag{20}$$

and

$$\left.\begin{aligned} &K(M) \sim 0{\cdot}3941, \qquad && Ac \sim n^{-1/2}M, \\ &\frac{A}{A_0} \sim \left(\frac{M_0}{M}\right)^n, && c \sim n^{-1/2}\frac{M^{n+1}}{A_0 M_0^n}, \\ &\int_{M_0}^{M} \frac{dM}{Ac} \sim n^{1/2}\log\frac{M}{M_0}, && \\ &\text{where } n = 2/K(\infty) = 5{\cdot}0743, && \end{aligned}\right\} \text{ as } M \to \infty. \tag{21}$$

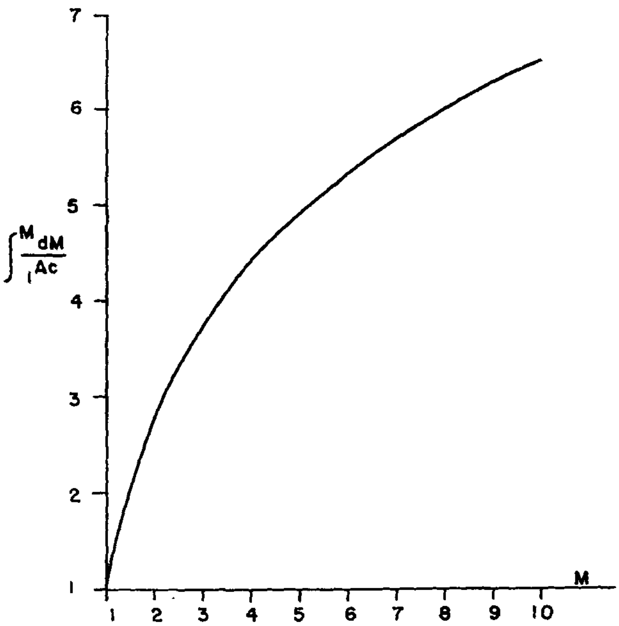

Figure 8. Graph of the function $\int_1^M dM/Ac$.

3. Diffraction of plane shocks

We consider a shock moving along a wall of given shape, specified by giving the inclination θ_w as a function of the distance s along it. We suppose that the wall is straight up to a certain point and that the shock is initially moving with constant Mach number M_0. The foot of the shock must always be normal to the wall; hence, the wall is a ray. If the wall is taken as $\beta = 0$, then θ is given on $\beta = 0$. *Provided that no shock–shocks occur,* the solution to this problem is the simple wave discussed in the previous section. Actually θ_w is given here as a function of s whereas we require its values in terms of α. These can be found, however. For, on $\beta = 0$

$$\alpha = \int \frac{ds}{M_w},$$

and M_w is determined from θ_w by the simple wave relation (see equation (14))

$$\theta_w = \int_{M_0}^{M_w} \frac{dM}{Ac}.$$

Expansion round a convex corner

For the special case of a convex corner, θ_w jumps from zero to a negative value and the solution is a centred simple wave. In the (α, β) plane, the characteristics (16) for the disturbed region form a fan, since they all start at the same point on $\beta = 0$. The equation of each characteristic is $\beta = \alpha\, c(M)$, and since α_w no longer appears this relation determines M immediately as a function of β/α. The shape of the shock will be as shown in figure 9; the radial lines shown here correspond to the characteristics and on each of them M is constant. The first disturbance spreads out on the shock at a rate given by $d\beta/d\alpha = c(M_0)$. If we choose β as the value of the distance y from the wall in the initial undisturbed motion so that $A_0 = 1$, then the actual speed of the first disturbance is $a_0\, c(M_0)$ (since $\alpha = a_0 t$).

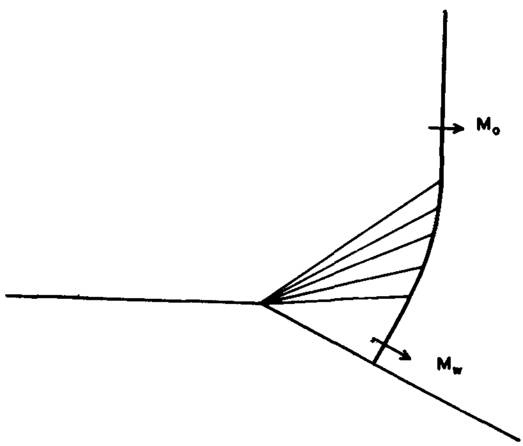

Figure 9. Diffraction of a shock around a convex corner.

Now for a small bend in the wall, i.e. for θ_w small, we can compare the results with the linear theory given Lighthill (1949). To be precise we compare the values predicted for the Mach number at the wall, M_w, and for the speed of propagation of the disturbance. For small θ_w, (14) reduces to

$$M_w - M_0 = Ac(M_0)\theta_w = \theta_w \{\tfrac{1}{2}(M_0^2 - 1)K(M_0)\}^{1/2}, \qquad (22)$$

(see (19)). We compare this with Lighthill's value in two extreme cases, $M_0 \to 1$ and $M_0 \to \infty$. For weak shocks, (22) gives

$$M_w - M_0 \sim \theta_w \{\tfrac{1}{2}(M_0 - 1)\}^{1/2}$$

whereas Lighthill has $8/(3\pi)$ times this. For strong shocks, (22) gives

$$M_w - M_0 \sim 0{\cdot}4439 M_0 \theta_w;$$

Lighthill's value has to be taken from a graph and all we can say is that the numerical factor is rather less than 0·5. In view of the relative simplicity of the derivation of (22) the results are remarkably good.

Turning now to a comparison of the speeds of propagation, it follows from the theory of sound that the first possible disturbance travels out in the flow behind the shock with the local sound speed a relative to the flow velocity u. Therefore the disturbance travels along the shock with speed

$$\{a^2 - (U-u)^2\}^{1/2}, \tag{23}$$

where U is the undisturbed shock velocity. The quantities U, a, u can all be expressed in terms of M_0 and it is found that (23) is $a_0 c^*$, where

$$c^* = \left[\frac{(M_0^2 - 1)\{(\gamma - 1)M_0^2 + 2\}}{(\gamma+1)M_0^2}\right]^{1/2}. \tag{24}$$

This is to be compared with the speed $c_0 \equiv c(M_0)$ given by (19). Since $A_0 = 1$, c_0 is the same as the graph of Ac in figure 7; the graph of c^* is also shown in figure 7. For weak shocks,

$$c_0 \sim \{\tfrac{1}{2}(M_0 - 1)\}^{1/2}, \qquad c^* \sim \{2(M_0 - 1)\}^{1/2}; \tag{25}$$

for strong shocks

$$c_0 \sim 0{\cdot}4439 M_0, \qquad c^* = 0{\cdot}4083 M_0. \tag{26}$$

Thus, the values are in reasonable agreement for shocks with $M_0 > 2$, say; but, for very weak shocks $c_0 = \tfrac{1}{2}c^*$. This discrepancy arises because the present theory cannot avoid concentrating the change in M over a relatively small part of the shock. For the stronger shocks, Lighthill's work shows that this concentration is correct; in fact he finds that the curvature becomes infinite as $M_0 \to \infty$. But for weak shocks the disturbance should be spread out over the whole of that part of the shock which is inside the sonic circle. The present theory compromises by concentrating the disturbance halfway to the sonic circle. Generally speaking then, the theory is more suitable for moderately strong shocks. However, the correct prediction of the value of $M_0 - M_w$ even in the weak case should not be ignored.

When the magnitude of θ_w is not small, M_w must be found from the exact form of (14) using the graph shown in figure 8. The solution is the centred simple wave determined from $c(M) = \beta/\alpha$, $c(M_w) < \beta/\alpha < c(M_0)$. As an example, the details of the solution are given for an infinitely strong shock. First of all, from (14) and (21), the Mach number at the wall is given by

$$M_w = M_0 \exp(\theta_w/\surd n). \tag{27}$$

Then the region $c(M_w) < \beta/\alpha < c(M_0)$ covered by the simple wave can be written

$$\exp\{(n+1)\vartheta_w/\sqrt{n}\} < \beta\sqrt{n}/\alpha M_0 < 1.$$

In the simple wave (using the results in (21))

$$c(M) = \frac{M^{n+1}}{\sqrt{n}M_0^n} = \frac{\beta}{\alpha} \quad \text{and } \theta = -\sqrt{n}\log\frac{M_0}{M}.$$

Therefore,

$$\left.\begin{aligned} \frac{M}{M_0} &= \left(\frac{\beta\sqrt{n}}{\alpha M_0}\right)^{1/(n+1)} \\ \theta &= \frac{\sqrt{n}}{n+1}\log\frac{\beta\sqrt{n}}{\alpha M_0} \end{aligned}\right\}, \quad \exp\{(n+1)\theta_w/\sqrt{n}\} < \frac{\beta\sqrt{n}}{\alpha M_0} < 1. \tag{28}$$

Along the shock, $\partial x/A\partial\beta = -\sin\theta$, $\partial y/A\partial\beta = \cos\theta$; therefore, at time $t = \alpha/a_0$, the shock is given in terms of the parameter β by

$$\left.\begin{aligned} x &= \alpha M_w \cos\theta_w - \int_0^\beta (M_0/M)^n \sin\theta\, d\beta, \\ y &= \alpha M_w \sin\theta_w + \int_0^\beta (M_0/M)^n \cos\theta\, d\beta. \end{aligned}\right\} \tag{29}$$

It should be noted that $x/M_0\alpha$ and $y/M_0\alpha$ are functions of the single quantity $\beta/M_0\alpha$, so that the shock pattern expands uniformly with time and a change in M_0 involves only a change of scale. The values of x and y in the simple wave are most easily calculated from (29) with θ as the parameter instead of β; they are

$$\left.\begin{aligned} \frac{x}{M_0\alpha} &= \frac{(n+1)^{1/2}}{n^{1/2}} e^{\theta/\sqrt{n}} \sin(\lambda-\theta), \\ \frac{y}{M_0\alpha} &= \frac{(n+1)^{1/2}}{n^{1/2}} e^{\theta/\sqrt{n}} \cos(\lambda-\theta), \end{aligned}\right\} \theta_w \leqslant \theta \leqslant 0 \tag{30}$$

where $\tan\lambda = \sqrt{n}$. The shape of the shock is plotted in figure 10 for the special case $\theta_w = -\frac{1}{2}\pi$.

For weaker shocks there is a limit on the magnitude of θ_w. For, M_w cannot decrease below unity; hence, if θ_w decreases below the value θ_{lim} given by

$$\theta_{\text{lim}} = \int_{M_0}^1 \frac{dM}{Ac},$$

the solution breaks down. Presumably this corresponds to separation of the flow at the corner which is known to occur in certain cases.

Compression at a concave corner: diffraction by a wedge

For a concave corner, the solution given by this theory is a shock–shock separating two regions in which M and θ are constant (see figure 11); this is Mach reflection. Following the remarks in §2, the most accurate determination of the solution in this theory is the conventional three-shock intersection treatment. Therefore, the only question that need be discussed here is the extent to which the conventional treatment can be replaced by our weak shock–shock relations (17) and (18). We consider the values predicted by (17) and (18) for the angle, χ, between the wall and the line joining the corner to the triple point. When the triple point is at (α, β),

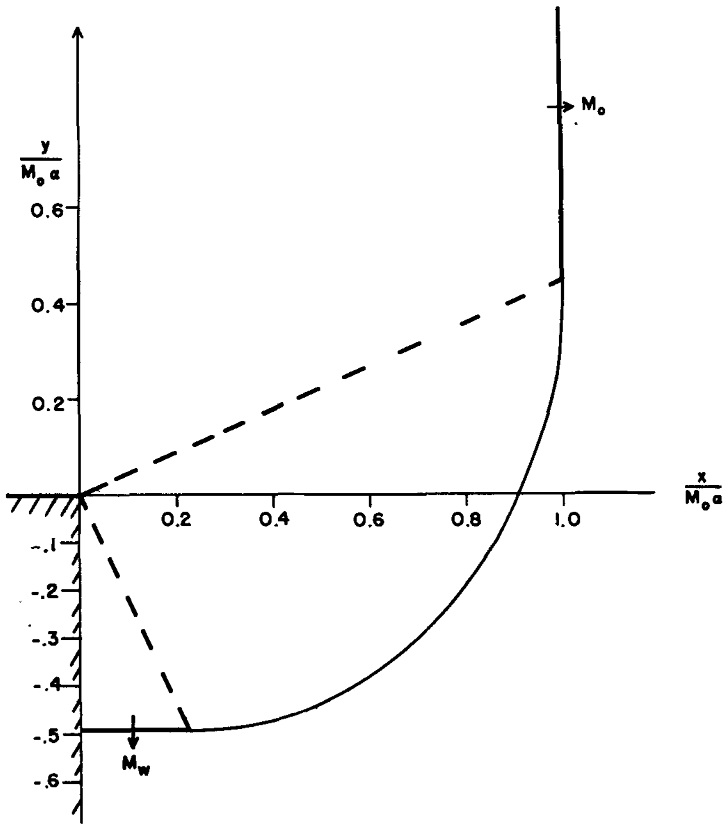

Figure 10. Diffraction of a strong shock around a right-angled corner ; shape of shock calculated from equation (30).

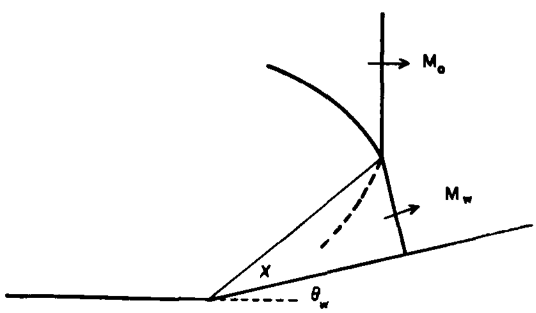

Figure 11. Diffraction of a shock by a wedge.

$\tan\chi = M_w\beta/A_w\alpha$, and β/α is equal to the shock–shock velocity C. Therefore, from (17),

$$\tan\chi = \frac{A_w}{A_0}\left\{\frac{1-(M_0/M_w)^2}{1-(A_w/A_0)^2}\right\}^{1/2} \tag{31}$$

Also, from (18),

$$\cot\theta_w = \frac{M_0}{M_w}\frac{1+A_wM_w/A_0M_0}{\{(1-M_0^2/M_w^2)(1-A_w^2/A_0^2)\}^{1/2}} \tag{32}$$

For strong shocks, the curve of χ against θ_w becomes independent of M_0, since A_w/A_0 is a function of M_w/M_0 (equation (26)). This curve is drawn in figure 12 and compared with the corresponding curve obtained from the three-shock theory (assuming that the Mach shock is approximately straight). We expect our shock–shock relations to apply when θ_w is small

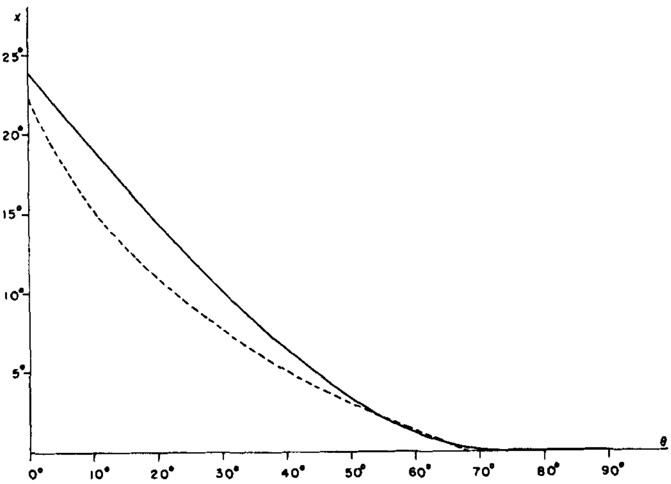

Figure 12. Values of χ and θ_w for the diffraction of a strong shock by a wedge. The full line refers to the present theory ; the broken line refers to the three-shock theory.

but to diverge ultimately from the more accurate value. It is seen that the error for small θ_w is about as expected, being of the same order as the discrepancy in (24). Then, fortuitously, the curves come closer together and actually cross. However, in the three-shock theory there is an upper limit on θ_w at which Mach reflection goes over into regular reflection, while the simple shock–shock relations become useless as they continue to predict Mach reflection. It is perhaps worth noting that if A_w is related to M_w by Laporte's formulae for a finite change in channel section, a cut-off value for θ_w is obtained (the value being $\theta_w = 33{\cdot}6°$). But as explained in § 2, this is not the correct direction in which to look for more accurate shock–shock conditions.

As an example of a shock of moderate strength the graph of χ against θ_w was found for $M_0 = 2{\cdot}42$, corresponding to a pressure ratio 20/3. This is shown in figure 13 and compared there with the experimental values quoted in Bleakney & Taub (1949).

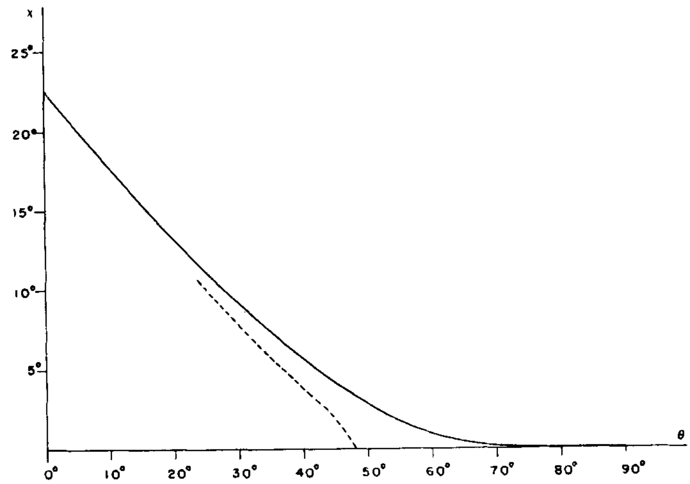

Figure 13. Comparison of the graph of χ against θ_w with the experimental results of Bleakney and Taub (indicated by broken line) in the case $M_0 = 2{\cdot}42$.

Wall of arbitrary shape

For the general problem, the solution must be obtained as in the analogous problems of gas dynamics; the slope of the wall corresponds to the velocity of the 'piston' in gas dynamics. If shock–shocks are not formed the solution is the simple wave described by (15) and (16) of § 2. Weak shock–shocks can be fitted in by a technique developed for gas dynamics and supersonic flow. They are formed when the characteristics overlap leading to a multivalued solution in (15), (16), because for points (α, β) in the overlapping region there will be more than one value of α_w. This is avoided by fitting in a shock–shock, according to the relations (17), (19). The rule (Whitham 1952) for fitting in a weak shock–shock is obtained in terms of the values of α_w on pairs of characteristics which meet at the shock–shock. Let the two values of α_w for such a pair of characteristics be α_1 and α_2. Then, the values of α_1 and α_2 are calculated for each β from the two relations

$$\beta = \frac{\alpha_2 - \alpha_1}{F(\alpha_2) - F(\alpha_1)} \tag{33}$$

$$\int_{\alpha_1}^{\alpha_2} F(\alpha)\, d\alpha = \tfrac{1}{2}(\alpha_2 - \alpha_1)\{F(\alpha_2) + F(\alpha_1)\}, \tag{34}$$

where $F(\alpha)$ is the function

$$1/c_0 - 1/c(M_w(\alpha)).$$

For the derivation of this type of result and for detailed explanations of the method of using it see Whitham (1952) (especially the Appendix). It is consistent with the approximations made already to approximate $F(\alpha)$ by the expressions

$$F(\alpha) \doteqdot \frac{c(M_w(\alpha)) - c_0}{c_0^2} \doteqdot \{M_w(\alpha) - M_0\} c_0^{-2} \left[\frac{dc}{dM}\right]_{M=M} \tag{35}$$

At each line $\beta =$ constant in the (α, β) plane, all the characteristics with their starting points α_w in the range $\alpha_1 < \alpha_w < \alpha_2$ are omitted since they have already been cut off by the shock–shock. In this way all except one of the values α_w given by (16) will be excluded, and the solution becomes single-valued. At the shock–shock, F jumps from the value $F(\alpha_1)$ to the value $F(\alpha_2)$; the corresponding jump in M is then given by (35). Certain applications of these results will be made in the next section.

Motion of a shock between two walls

The motion of a shock between two walls may also be treated by this theory. It would be analogous to the problem in gas dynamics of the waves produced in a tube of finite length by pistons in each end of the tube. The given shape of the walls corresponds to given motions of the pistons. In this case waves would move in each direction across the shock face, being reflected from each wall in turn. The solution would require numerical computations (by the method of characteristics, say).

4. Stability of plane shocks

It is well known that plane shocks are stable; that is, if the shock is disturbed slightly from the plane shape the disturbance decays as the shock propagates. This property is now investigated using the theory of wave propagation on the shock, and the rate at which a disturbance will die out is determined.

As a special case we may consider the disturbances generated as the shock moves along a wall, and the results of the last section may be used to study the decay of the disturbance. Let us take first the case of a small bump on the wall. Then $\theta_w = 0$ except on a finite length of the wall. In this case it can be shown that after a sufficient time the disturbance will become an 'N-wave', i.e. there is a shock–shock at both the head and tail of the disturbance. The maximum values of θ and $M - M_0$ are taken at the shock–shock, so it is sufficient to calculate the rate at which these values decay. Since the wave is moving into an undisturbed part of the shock, $F(\alpha_1) = 0$ in (33) and (34), and therefore

$$\beta = \frac{\alpha_2 - \alpha_1}{F(\alpha_2)}, \qquad \int_0^{\alpha_2} F(\alpha)\, d\alpha = \tfrac{1}{2}(\alpha_2 - \alpha_1) F(\alpha_2).$$

On eliminating $(\alpha_2 - \alpha_1)$, we have the relation

$$\beta = \frac{2}{F^2(\alpha_2)} \int_0^{\alpha_2} F(\alpha)\, d\alpha$$

to determine the characteristic variable α_2 behind the shock in terms of β. For large β it is clear that the corresponding α_2 tends to the zero of F. If this zero is denoted by α_0 we have

$$F(\alpha_2) \sim \left\{\frac{2}{\beta}\int_0^{\alpha_2} F(\alpha)\,d\alpha\right\}^{1/2}. \tag{36}$$

The changes in M and θ at the shock–shock are proportional to $F(\alpha_2)$: from (35),

$$M - M_0 = \left[c^2 \frac{dM}{dc}\right]_{M=M_0} F(\alpha_2),$$

and, from (13) with $A_0 = 1$, we have

$$\theta = (M - M_0)/c_0.$$

Thus, the disturbance decays like $\beta^{-\frac{1}{2}}$ as it moves away from the wall, and since the disturbance is weak, β is approximately equal to the distance from the wall.

Blackburn (1953) investigated the case of a shock moving along a sinusoidal wall using Lighthill's linear theory. According to the present theory the sinusoidal variation sends out a series of successive compression and expansion waves, and the compression waves eventually break to form a series of shock–shocks. As $\beta \to \infty$, the values of α_1 and α_2 given in (33) approach successive zeros of $F(\alpha)$, say α_0 and $\alpha_0 + l$, and we see from (33) that the jump in $F(\alpha)$ across each shock is given by

$$\Delta F = -l/\beta.$$

In place of l, we may introduce the wavelength λ of the sinusoidal wall which is approximately lM_0; also $\beta \doteqdot y$. Then the changes in M and θ are given in terms of ΔF as before, and we have

$$\Delta M = \frac{c_0^2}{M_0}\left[\frac{dM}{dc}\right]^{-1}_{M=M_0} \frac{\lambda}{y}, \qquad \Delta\theta = \frac{c_0}{M}\left[\frac{dM}{dc}\right]_{M=M_0} \frac{\lambda}{y}. \tag{37}$$

It is interesting to note that these values are independent of the amplitude of the sinusoidal variations in the wall slope. The factors multiplying λ/y in (37) are increasing functions of M_0 so that the stability decreases with increasing Mach number.

Now the decay like λ/y predicted here does not agree with Blackburn's results. He finds that the decay is proportional to $(\lambda/y)^{3/2}$ in general, but as $M \to \infty$ the law changes over to decay proportional to $(\lambda/y)^{1/2}$. But the mechanisms of decay in the two theories are completely different. In the present theory, non-linearity is essential; the decay is brought about by the shock–shocks. Indeed if the theory were linearized in the usual way, the waves would not decay at all. On the other hand, Blackburn's theory is a linear one and he finds a decay due to a damping of the waves which does not appear here. If we pursue the analogy to gas dynamics, the present theory corresponds to the non-linear theory of sound waves, neglecting viscosity and heat conduction except where their effects are concentrated in shocks; the Blackburn theory corresponds to the linear theory of acoustics with viscosity and heat conduction included. However, it would be dangerous to push this analogy too far, and it should not be used in assessing the relative importance of the two mechanisms of stability.

Another formulation of the stability problem which leads to similar results is the initial value problem in which the shape and Mach number of the shock are given at some time, i.e. θ and M are prescribed functions of β on $\alpha = 0$. The problem of a non-plane piston moving forward with uniform velocity into a gas at rest is a special case of this. In this case waves move in each direction on the shock. If the shock is initially plane and uniform except in a section of finite length, there will be an interaction region at first, but eventually the disturbance will separate into two simple waves one moving in each direction. Each of these simple waves is similar to the solution obtained for the problem of a bump on the wall, and decays like $t^{-1/2}$. An accurate solution in the interaction region could be obtained numerically using the method of characteristics. But if the disturbance is not too large and the interaction region not too wide, the linearized form of the solution will give quite good results and it can be given explicitly. Let us, for example, consider the case in which M takes the constant value M_0 on $\alpha = 0$ and θ is a given function of β. In the linearized theory, (7) and (8) are approximated by

$$\frac{\partial \theta}{\partial \beta} + \frac{1}{c_0^2 A_0} \frac{\partial M}{\partial \alpha} = 0, \tag{38}$$

$$\frac{\partial \theta}{\partial \alpha} + \frac{1}{A_0} \frac{\partial M}{\partial \beta} = 0. \tag{39}$$

Then, θ and M both satisfy the wave equation and we have the solution

$$\theta = g_1(\beta - c_0 \alpha) + g_2(\beta + c_0 \alpha),$$

where g_1 and g_2 are arbitrary functions. (This solution corresponds to approximating the characteristics in the exact theory by lines $\beta \pm c_0 \alpha$.) Since M is a constant on $\alpha = 0$, (39) shows that $\partial\theta/\partial\alpha = 0$. Therefore the functions g_1 and g_2 are to be determined from initial conditions $\theta = \theta_0(\beta)$, say, and $\partial\theta/\partial\alpha = 0$ on $\alpha = 0$. These determine the solution as

$$\theta = \tfrac{1}{2}\{\theta_0(\beta - c_0 \alpha) + \theta_0(\beta + c_0 \alpha)\}. \tag{40}$$

The curvature of the shock is proportional to $\partial\theta/\partial\beta$ and so satisfies a similar rule. This shows very directly the tendency of the curvature of the shock to be averaged out. For example, the curvature never exceeds the maximum value of the initial curvature. Thus even the linear theory rectifies the failure of geometrical acoustics near a caustic, and the curvature remains bounded. The linear theory is not uniformly valid as α becomes large due to the divergence of the characteristics, and we must go over to the accurate determination.

Freeman (1955) considers the special case of a shock which is initially sinusoidal in shape, and as would be expected his results for the decay of the disturbance are similar to Blackburn's. In our theory we have a rather complicated situation with shock–shocks moving in each direction and interacting with each other. But the results will be essentially the same as (37) with the disturbance decaying like $1/t$.

As a preliminary investigation Freeman considers the shock produced by a nearly plane wedge-shaped piston moving forward with uniform velocity. It is perhaps worth noting our solution for this case. The initial Mach number M_0 of the shock is constant, and is easily found in terms of the piston velocity from the shock conditions. The initial values of θ are $\theta_0 = -\delta$ in $\beta < 0$, $\theta_0 = +\delta$ in $\beta > 0$, where $\pi - 2\delta$ is the angle of the wedge.

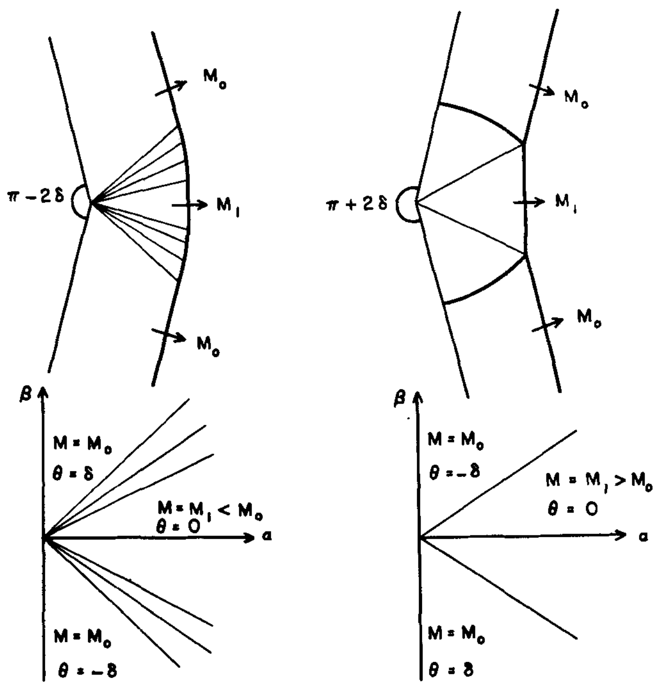

Figure 14. Shock patterns produced by wedge pistons.

If the wedge is convex forward, the solution is given by two centred simple waves. Through the simple wave moving in the direction of β increasing

$$\theta - \delta = \int_{M_0}^{M} \frac{dM}{Ac},$$

and through the other wave

$$\theta + \delta = -\int_{M_0}^{M} \frac{dM}{Ac}.$$

In between the simple waves, both these relations hold; hence, $\theta = 0$ and the Mach number M_1 is determined by

$$2\delta \doteq -\int_{M_0}^{M_1} \frac{dM}{Ac} \doteq \frac{M_0 - M_1}{A_0 c_0}. \tag{41}$$

If the wedge is concave forward, δ is negative and the solution is given by two shock–shocks. The physical plane and the (α, β) plane are shown for each case in figure 14.

Although these solutions have been introduced for a rather artificial problem, they also furnish results for the more interesting problem of reflection of a plane shock from a nearly plane wall. It is only necessary to choose the frame of reference so that the air behind the incident shock is at rest. Then the reflected shock is given by the above solution; the flow pattern for both a convex and a concave corner are shown in figure 15.

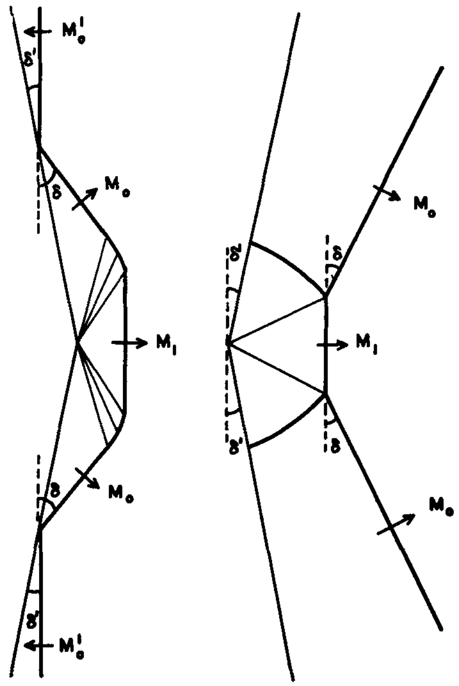

Figure 15. Reflection of shocks at wedge-shaped walls.

The appropriate values of δ, M_0, a_0 are obtained from the Mach number M' of the incident shock, the wedge angle $\pi - 2\delta'$, and the sound speed a' ahead of the incident shock, using the well-known results for regular reflection. When the incident shock moves with Mach number M' into air at rest, the particle velocity u' behind it is given by

$$\frac{u'}{a'} = \frac{2}{\gamma+1}\left(M' - \frac{1}{M'}\right).$$

The velocity $-u'$ is superimposed on the flow in order to obtain the required frame of reference. In this frame, the reflected shock moves into air at rest and it can be described by the above theory. The sound speed a_0 is given by

$$\frac{a_0^2}{a'^2} = \frac{\{2\gamma M'^2 - (\gamma-1)\}\{(\gamma-1)M'^2+2\}}{(\gamma+1)^2 M'^2},$$

and the Mach number M_0 and the inclination δ of the 'undisturbed part' of the reflected shock are given by

$$\delta = \left\{\frac{3\gamma-1}{\gamma+1} + \frac{3-\gamma}{(\gamma+1)M'^2}\right\}\delta', \qquad M_0 = \frac{a'M'}{a_0}\left\{\frac{2\gamma}{\gamma+1} - \frac{\gamma-1}{(\gamma+1)M'^2}\right\}.$$

5. Stability of a converging cylindrical shock

In contrast to plane shocks, it is found that converging cylindrical and spherical shocks are unstable. As noted in § 1, a theoretical demonstration of this instability has been given by Butler (1955). We now give a somewhat improved presentation of Butler's theory which avoids making the small perturbation approximation.

A converging cylindrical shock will ultimately become strong, so that for the stability investigation it is sufficient to consider the case of strong shocks. We suppose that the Mach number is constant and equal to M_0 at some initial time, and we choose β to be the distance of the ray along the initial position of the shock so that $A_0 = 1$. Then we take

$$A = (M_0/M)^n, \qquad n = 5{\cdot}0743,$$

as given in (26), and the equations (7) and (8) for θ and M become

$$\frac{\partial\theta}{\partial\beta} + \frac{nM_0^n}{M^{n+2}}\frac{\partial M}{\partial\alpha} = 0, \tag{42}$$

$$\frac{\partial\theta}{\partial\alpha} + \frac{M^n}{M_0^n}\frac{\partial M}{\partial\beta} = 0. \tag{43}$$

For a shock with cylindrical symmetry and initial radius R_0, $\theta = -\beta/R_0$ and M is a function of α. From (42), we see that

$$M = M_0\left(-\frac{(n+1)M_0\alpha}{nR_0}\right)^{-1/(n+1)}, \tag{44}$$

α being chosen so that it reaches the value zero when the shock gets to the centre. This is Chisnell's approximate form of the exact Guderley solution (Guderley 1942). For $\gamma = 1{\cdot}4$, $(n+1)^{-1} = 0{\cdot}16463$; the corresponding exponent in Guderley's solution (calculated with great accuracy by Butler (1954)) is $0{\cdot}16478$. In his discussion of stability, Butler chooses the value of n from Guderley's solution.

Now the question is whether small deviations in the initial shape of the shock increase or decrease as the shock contracts. The hodograph transformation will be used to investigate this question. First it is convenient to introduce new variables:

$$q = (M/M_0)^{n+1}, \qquad \chi = (n+1)\theta/\sqrt{n}, \qquad s = M_0\alpha/\sqrt{n}.$$

Then (42) and (43) take the neater form

$$\frac{\partial q}{\partial s} + q^2\frac{\partial\chi}{\partial\beta} = 0, \tag{45}$$

$$\frac{\partial\chi}{\partial s} + \frac{\partial q}{\partial\beta} = 0. \tag{46}$$

The symmetrical solution is the one in which $q \propto 1/s$, $\chi \propto \beta$. (It may be noted in passing that another exact solution is

$$q = \beta/s, \qquad \chi = \log(\beta/s);$$

this describes the motion of a shock down a certain curved channel.) In the hodograph transformation, the roles of the dependent and independent variables are interchanged; the transformation formulae for the derivatives are

$$\chi_\beta = Js_q, \qquad \chi_s = -J\beta_q, \qquad q_\beta = -Js_\chi, \qquad q_s = J\beta_\chi$$

where the Jacobian $J = \partial(q,\chi)/\partial(s,\beta)$. On substituting these expressions in (45) and (46), we have the *linear* equations

$$\beta_\chi + q^2 s_q = 0, \qquad \beta_q + s_\chi = 0.$$

It is convenient to eliminate β to get the single equation

$$q^2 s_{qq} + 2qs_q = s_{\chi\chi}.$$

Solutions to this equation are

$$s = q^\mu e^{im\chi},$$

where

$$\mu = -\tfrac{1}{2} \mp (\tfrac{1}{4} - m^2)^{1/2}.$$

If $m = 0$, $\mu = -1$ gives the symmetrical solution. If $m \geqslant 1$, μ is complex with $\mathscr{R}(\mu) = -\frac{1}{2}$. Therefore, when $q \to \infty$ (as the shock contracts to the centre), the harmonics eventually dominate the symmetrical mode. *Hence the shock is unstable.*

The appearance of an imaginary part in μ is also of some interest since it shows that the disturbance again takes the form of waves travelling round the shock. When the disturbance grows large, it is possible for the Jacobian J to vanish. This means that the mapping from the hodograph plane (q,χ) to the (s,β) plane ceases to be single-valued, and so-called limit lines appear. These correspond to the formation of shock-shocks. When this stage is reached, further calculations of the motion would be more easily carried out directly in the (s,β) plane by numerical methods.

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